A SMALL ULTRAFILTER NUMBER AT EVERY SINGULAR CARDINAL

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Abstract. We obtain a small ultrafilter number at $\aleph_{\omega_1}$. Moreover, we develop a version of the overlapping strong extender forcing with collapses which can keep the top cardinal κ inaccessible. We apply this forcing to construct a model where κ is the least inaccessible and V_κ is a model of GCH at regulars, failures of SCH at singulars, and the ultrafilter numbers at all singulars are small.

1. introduction

Some of the most basic mathematical theorems rely on the possibility to distinguish between *small* and *large* sets. For example, the Lebesgue criteria for Riemann integratability states that a bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable, if and only if the set of its discontinuity points is *small*, which in this case means of Lebesgue measure zero. Smallness has many other interpretations: small cardinalities in set theory, nowhere dense sets in topology, probability zero events in probability theory, or polynomial and linear functions in computability theory. An abstract approach to define a notion of largeness for the subsets of a given set X is *filters*, which is simply a set $F \subseteq P(X)$ that contains all the large subsets of X. Formally speaking, we require the following 3 axioms which say that F is a *filter over* X:

(1) $X \in F$, $\emptyset \notin F$. (non empty and non-degenerate)
(2) $A, B \in F \Rightarrow A \cap B \in F$. (closed under intersection)
(3) $(A \in F \wedge A \subseteq B) \Rightarrow B \in F$. (upward closed to inclusion)

For a fixed filter F, we may consider small sets as the sets whose complements are in F. Note that for many filters, there are sets X which are neither small, nor large, namely $X \notin F$ and also $X^c \notin F$ e.g. in probability, there are sets X such that $0 < \mathbb{P}(X) < 1$ and thus are neither small (i.e. $\mathbb{P}(X) = 0$), nor large (i.e. $\mathbb{P}(X) = 1$).

Ultrafilters are those filters which do determine that every set is either large or small. Namely, a filter U over X is an *ultrafilter* if for every $B \subseteq X$ either $B \in U$ or $X \setminus B \in U$. Most of the non-trivial examples of ultrafilters involve the Axiom of choice and are thus highly non-constructive. However,

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they have been proven to be useful in many areas such as Analysis, Topology, Model theory, Algebra, and combinatorics. For example, Nonstandard analysis [13] is an alternative approach to study analysis and more sophisticated mathematics. The ϵ-δ definitions in analysis can be replaced by using more concrete objects, so-called infinitesimals, in a nonstandard universe of nonstandard reals, which contain all reals. One of the constructions of a nonstandard universe is through an ultraproduct construction, which requires a non-trivial ultrafilter over $\mathbb{N}$. there is a more concrete and the Stone-Čech compactification in topology [21],[3]. Studying the combinatorial nature of ultrafilters is important to obtain a stronger understanding of those applications, but are not limited to that and can be used in several results in infinitary combinatorics (see for example [15]). One specific combinatorial property we are interested in this paper is the ultrafilter number, which has been extensively studied in recent years, as we will see in the next subsection:

1.1. **The ultrafilter number.**
The ultrafilter number for a cardinal number κ, determines how many sets one needs in order to generate an ultrafilter on κ. Let us be more precise here:

Definition 1.1. Let U be an ultrafilter over a cardinal κ, define:

(1) a subset of an ultrafilter U, $\mathcal{B} \subseteq U$ is called a *base* if $\forall A \in U$ there is $B \in \mathcal{B}$ such that $B \subseteq^* A$[1].
(2) The *characteristics of* U is $Ch(U) := \min(|\mathcal{B}| \mid \mathcal{B}$ is a base for$U)$
(3) The ultrafilter number $\mathfrak{u}_\kappa := \min(Ch(U) \mid U$ is a uniform ultrafilter over $\kappa)$[2]

The number $\mathfrak{u}_\kappa$ is a *generalized characteristic cardinal of the continuum* as it is known that for every κ, $\kappa^+ \leq \mathfrak{u}_\kappa \leq 2^\kappa$. As with other characteristic cardinals, the basic question is whether they can be (namely, is it consistent to) separated from the continuum. Kunen (Exercise (A10) of Chapter XIII in [16]) proved that using a suitable iteration (of Mathias forcing) over a model of CH, one can force a model with $2^{\aleph_0} > \mathfrak{u}_{\aleph_0}$. Kunen's method does not generalize to greater cardinals, and raised whether it is consistent to have $\mathfrak{u}_{\aleph_1} < 2^{\aleph_1}$. This question is open and has been so since the 70s.

Assuming stronger assumptions on the cardinal κ, which are known as *large cardinal assumptions*, Gitik and Shelah [12, Lemma 1.9] forced the existence of a cardinal $\aleph_0 < \kappa$ with $2^\kappa > \mathfrak{u}_\kappa$. This cardinal κ is extremely greater than ω_1 and do not lay near the area where other mathematics occurs. The large cardinal assumption was then improved by Brooke-Taylor, Fischer, Friedman, and Montoya [2] to a supercompact cardinal, where they used a similar iteration to the one Kunen used, but still considered an extremely large cardinal. Recently, a remarkable result of Raghavan and Shelah [20] established the consistency of $\mathfrak{u}_\kappa < 2^\kappa$ for $\kappa = 2^{\aleph_0}$ where they started with a much smaller large cardinal- a *measurable cardinal.* However, in their model

[1] The order $A \subseteq^* B$ is defined by $A \setminus B$ is bounded in κ.

[2] An ultrafilter U over κ is uniform if for every $X \in U$, $|X| = \kappa$.

$2^{\aleph_0}$ is still very large. They also obtained the result on the much smaller cardinal $\aleph_{\omega+1}$ but starting again from a supercompact cardinals.

While all the work mentioned above concerns the ultrafilter number for regular cardinals, a different line of research about the ultrafilter number on singular cardinals has also been studied in the last decade. The first results in this direction uses PCF theory and is due to Shelah and Garti [9, Theorem 1.4]:

Theorem 1.2. *Suppose that $\kappa = cf(\lambda) < \lambda$ are two cardinals, λ is strong limit, and $\langle \lambda_i \mid i < \kappa \rangle$ is an unbounded and increasing sequence in λ such that:*

(1) *There is a uniform ultrafilter E over κ.*
(2) *Each U_i is a uniform ultrafilter over λ_i carrying a strong-base $\langle A_{i,\beta} \mid \beta < \theta_i \rangle$.* [3] *Then there is a uniform ultrafilter U over λ such that $Ch(U) \leq tcf(\prod_{i<\kappa} \lambda_i, <_E) \cdot tcf(\prod_{i<\kappa} \theta_i, <_E)$*

This theorem provides a way to construct a uniform ultrafilter with a small base, and one can apply in is various models where these tcf's have known values. For example, for any fixed κ, Garti and Shelah [8] have a model where $2^\lambda > \lambda^+$, a sequence of measurables $\langle \lambda_i \mid i < \kappa \rangle$, $2^{\lambda_i} = \lambda_i^+$ and $tcf(\prod_{i<\kappa} \lambda_i, <_E) = tcf(\prod_{i<\kappa} \lambda_i^+, <_E) = \lambda^+$. The measures U_i can be any normal ultrafilter over λ_i, then by normality and $2^{\lambda_i} = \lambda_i^+$ when can get $\theta_i = \lambda_i^+$. So they obtained the following:

Theorem 1.3. *Suppose that there is a supercompact cardinal, then there is a model with a singular cardinal λ such that $\mathfrak{u}_\lambda = \lambda^+$ and $2^\lambda > \lambda^+$.*

Then Garti, Magidor and Shelah [7] used the single extender-based forcing to get the following:

Theorem 1.4. *Suppose that $\kappa < \lambda$ are such that κ is strong and $\lambda > \kappa$ is a limit of measurable cardinals $\langle \lambda_i \mid i < \theta \rangle$. Then in the generic extension by the extender based forcing with E being a (κ, λ)-extender, for every $i < \theta$, there are ω-sequences of measurables $\langle \lambda_{i,n} \mid n < \omega \rangle$ (corresponding to the measure U_{λ_i}) in κ such that $tcf(\prod_{n<\omega} \lambda_{i,n}/J_{bd}) = \lambda_i^+$. $2^\kappa \geq \lambda$ and $GCH_{<\kappa}$. In particular, there are ultrafilters U_i over κ such that $Ch(U_i) = \lambda_i^+$.*

From the previous model we can put collapses and obtain other values of $Ch(U)$.

It is natural to use Magidor-Radin extender-based forcing of Merimovich [18] to push these results to uncountable cofinalities. Indeed, Cummings and Morgan [5] manneged to obtain it and proved the following:

Theorem 1.5. *Let $\rho < \kappa < \lambda$ where ρ is regular and uncountable, λ is the least inaccessible limit of measurable cardinals greater than κ, and there is a Mitchell increasing sequence $\langle E_i \mid i < \rho \rangle$ such that each extender E_i*

[3]Strong-base for U is sequence $\langle A_\alpha \mid \alpha < \theta \rangle$ which is $\subseteq^*$-decreasing, each $A_\alpha \in U$, and for every $B \in U$, exists $\alpha < \theta$ such that $A_\alpha \subseteq^* B$.

witnesses that κ is λ-strong and is such that ${}^{\kappa}Ult(V,E_i) \subseteq Ult(V,E_i)$. Then there is a cardinal-preserving generic extension in which $cf(\kappa)=\rho$, $2^{\kappa}=\lambda$, and $Sp_{\chi}(\kappa)$ is unbounded in λ.

Finally, Gitik Shelah and Garti [6], brought everything down to $\aleph_\omega$ and got the result for $\aleph_\omega$. In the first part of this paper we will use the forcing [14] to tackle the question of separating $\mathfrak{u}_{\aleph_{\omega_1}}$ and $2^{\aleph_{\omega_1}}$ and prove the following:

Theorem A. *Suppose κ is a singular cardinal, $\rho<\kappa$ is regular and $\langle \kappa_i \mid i<\rho\rangle$ is a sequence of strong cardinals with limit κ and $\rho<\kappa_0$. Suppose that $\vec{E}=\langle E_i \mid i<\rho\rangle$ is a Mitchell increasing sequence of extenders witnessing that κ_i is κ^{++}-strong, then after forcing $\mathbb{P}_{\vec{E}}$ we obtain a model where $\kappa=\aleph_{\omega_1}$, $2^{\kappa}>\aleph_{\omega_1+1}$ and $\mathfrak{u}_\kappa=\aleph_{\omega_1+1}$.*

The proof generalizes ideas similar to the one from [6], but also applies for the countable case. Also, we provide some missing details in the proof from [6].

1.2. **Overlapping Strong Extenders of long length.** As mentioned in the previous subsection, many results, both about the ultrafilter number and others, are known to hold at extremely large cardinals. One important task is to verify whether this results hold at more *down to earth* cardinals. This problem is a typical problem when we force with so-called *Prikry-type* forcings. The idea is to start with a large cardinal and to force a model where we destroy some properties of this large cardinal while other properties survive the forcing. This leads to solutions of many important open problems, such as the *Singular Cardinal Hypothesis* [17]. This type of forcing is in extensive use in modern set theory. The major disadvantage of such a forcing is that even though the initial large cardinal loses its large cardinal property, it might still be located in a very high spot of the mathematical universe, namely above many other large cardinals, and therefore, irrelevant to solve problems in the lower cardinals. Fortunately, a mechanism of crossing the gaps between the lower cardinals and the higher ones is also available in some situations. This is the so-called Prikry-type forcing with interleaving collapses, which both preserves the large cardinal properties and brings everything down to much lower cardinal (see for example, Chapter 4 of[10]). While Prikry-type forcings usually singularize a cardinal, some variations of the Radin forcing [19] and of Gitik's overlapping extender-based forcing [1] can keep κ regular. In Section 3, we show how to incorporate collapses with these kinds of forcings and develop the *Long Overlapping strong extender with collapses*, which is in the spirit of the one from [14] or prior to that of Gitik's overlapping extender based forcing with collapses. The main innovation of our forcing is that it can keep κ inaccessible, and by involving collapses, turn κ into the first inaccessible. This fact is relevant for those problems in set theory seeking for consistency results at the first inaccessible cardinals, and we believe that this forcing might be useful to tackle such problems. For example, we apply this forcing to obtain the following model:

Theorem B. *Let κ be the least inaccessible cardinal such that there is a Mitchell increasing sequence $\langle E_i \mid i < \kappa\rangle$ witnessing that each κ_i is κ^{++}-strong. Then it is consistent that κ is the least inaccessible cardinal, GCH holds for every regular below κ, SCH fails for every singular below κ and for $\lambda < \kappa$ singular, $\mathfrak{u}_\lambda < 2^\lambda$.*

Convention: $p \geq q$ means p is stronger than q. For functions f and g with $\mathrm{dom}(g) \subseteq \mathrm{dom}(f)$, define $f \oplus g$ (f *overwritten by* g) as the function h with $\mathrm{dom}(h) = \mathrm{dom}(f)$, $h(x) = g(x)$ for $x \in \mathrm{dom}(g)$ and $h(x) = f(x)$ otherwise.

2. A small ultrafilter number at $\aleph_{\omega_1}$

We start with a basic definition.

Definition 2.1. Let κ be an infinite cardinal and U is a uniform[4] ultrafilter on κ.

(1) A *base* for U is a collection $\mathcal{B} \subseteq U$ such that for every $B \in U$, there is $A \in \mathcal{B}$ such that $A \subseteq^* B$, namely, $A \setminus B$ is a bounded subset of κ.
(2) $Ch(U) := \min\{|\mathcal{B} \mid \mathcal{B} \subseteq U \text{ is a base for } U\}$
(3) The ultrafilter number of κ, denoted by $\mathfrak{u}_\kappa$, is

$$\min\{Ch(U) \mid U \text{ is a uniform ultrafilter}\}.$$

By Claim 1.2 of [9], for any infinite κ, $\kappa < \mathfrak{u}_\kappa \leq 2^\kappa$ and if $2^\kappa = \kappa^+$, then $\mathfrak{u}_\kappa = 2^\kappa$. It is possible to have $Ch(U)$ being singular [11]. We say that $\langle W, \leq_W\rangle$ is a *pre-order* if it is reflexive and transitive. The terms "dense" and "open" take their usual meanings. Let us use some of the definitions of Garti, Gitik, and Shelah from [6]:

Definition 2.2. Let $\langle W_i \mid i < \lambda\rangle$ be a sequence of pre-orders and F be a filter on λ. A *Sullam* in $(\prod_{i<\lambda} W_i, F)$ is a sequence $\langle f_\alpha \mid \alpha < \mu\rangle \subseteq \prod_{i<\lambda} W_i$ such that:

(1) $\langle f_\alpha \mid \alpha < \mu\rangle$ are increasing *modF*, namely, if $\alpha < \beta < \mu$, then

$$\{i < \lambda \mid f_\alpha(i) <_{W_i} f_\beta(i)\} \in F.$$

(2) for any list $\langle V_i \mid i < \lambda\rangle$ such that $V_i \subseteq W_i$ is open dense, there is $\alpha < \mu$ such that

$$\{i < \lambda \mid f_\alpha(i) \in V_i\} \in F.$$

We only focus on the dual filters of the bounded ideal $J_{bd} = \{X \subseteq \lambda \mid |X| < \lambda\}$. The collection of positive sets F^+ has the usual meaning, namely, if $I_F = \{\lambda \setminus X \mid X \in F\}$ is the dual ideal, then $F^+ := \{X \subseteq \lambda \mid X \notin I_F\}$.

Let F be a filter over λ and W be a pre-order. A function $g : W \to F^+$ is said to be *order preserving* when for every $p \leq_{W_i} q$, $g(q) \subseteq g(p)$. We say that g is *deciding* if for every $A \subseteq \lambda$ and any $w \in W$, there is $w \leq_W u$ such that $g(u) \subseteq A$ or $g(u) \subseteq \lambda \setminus A$.

[4] A filter over a cardinal κ is uniform iff it contains the Fréchet filter: $\{X \subseteq \kappa \mid |\kappa \setminus X| < \kappa\}$.

Definition 2.3. Given a singular cardinal $\lambda = \mathrm{cf}(\mu) < \mu$, a *nice system* $\mathcal{S}$ consists of the following data:

(1) A cofinal sequence $\langle \lambda_i \mid i < \lambda \rangle$ in μ consisting of regular cardinals.
(2) A sequence $\langle D_i \mid i < \lambda \rangle$ such that each D_i is a uniform filter over λ_i.
(3) A sequence $\langle W_i \mid i < \lambda \rangle$ of pre-orders.
(4) Functions $g_i : W_i \to D_i^+$ which are order-preserving and deciding.

The following is a slight variation of [6, Theorem 1.3]:

Theorem 2.4. *Suppose that $\lambda = cf(\mu) < \mu$, $\mathcal{S}$ is a nice system and D is a uniform ultrafilter over λ. Suppose that $\theta \in (\mu, 2^\mu]$ is a regular cardinal such that:*

(1) *$Ch(D) \leq \theta$.*
(2) *there is a Sullam $\langle f_\beta \mid \beta < \theta \rangle$ in $(\prod_{i<\lambda} W_i, D)$.*

Then there is a uniform ultrafilter U over μ such that $Ch(U) \leq \theta$.

Proof. The definition of U is as follows, for $X \subseteq \mu$:

$$X \in U \iff \exists \alpha < \theta \{ i < \lambda \mid g_i(f_\alpha(i)) \subseteq_{D_i} X \cap \lambda_i \} \in D,$$

where $\subseteq_{D_i}$ means $\subseteq^*$ with respect to the filter D_i.

Claim 2.5. *U is a uniform ultrafilter over μ.*

Proof of claim. First, note that since $\mathrm{rng}(g_i) = D_i^+$, $g_i(f_\alpha(i)) \neq \emptyset (mod D_i)$, hence if $X = \emptyset$, then $\{i < \lambda \mid g_i(f_\alpha(i)) \subseteq_{D_i} X \cap \lambda_i\} = \emptyset$. Since D is a uniform filter, by the definition of U, $\emptyset \notin U$. A similar argument shows that $\mu \in U$. If $X_1, X_2 \in U$, then there are $\alpha_1, \alpha_2 < \theta$ such that

$$E_l := \{ i < \lambda \mid g_i(f_{\alpha_l}(i)) \subseteq_{D_i} X_l \cap \lambda_i \} \in D, \quad \text{for } l = 1, 2.$$

Suppose without loss of generality that $\alpha_1 \leq \alpha_2$. Then by the definition of Sullam, the set

$$E_3 := \{ i < \lambda \mid f_{\alpha_1}(i) <_{W_i} f_{\alpha_2}(i) \} \in D.$$

Since all the g_i's are order-preserving, for every $i \in E_3$, $g_i(f_{\alpha_2}(i)) \subseteq g_i(f_{\alpha_1}(i))$. It follows that if $i \in E_1 \cap E_2 \cap E_3 \in D$, we have that

(1) $g_i(f_{\alpha_2}(i)) \subseteq_{D_i} X_2 \cap \lambda_i$ (since $i \in E_2$).
(2) $g_i(f_{\alpha_2}(i)) \subseteq g_i(f_{\alpha_1}(i)) \subseteq_{D_i} X_1 \cap \lambda_i$ (Since $i \in E_3$ and $i \in E_1$, resp.).

It follows that $g_i(f_{\alpha_2}(i)) \subseteq_{D_i} X_1 \cap X_2 \cap \lambda_i$. By the definition of U, we conclude that $X_1 \cap X_2 \in U$. Showing that U is closed upward is straightforward. To see it is an ultrafilter, let $X \subseteq \mu$. For every $i < \lambda$, consider the set

$$V_i = \{ q \in W_i \mid (g_i(q) \subseteq X \cap \lambda_i) \vee (g_i(q) \subseteq \lambda_i \setminus X) \},$$

then V_i is dense. Since g_i is order-preserving, V_i is also open. By the definition of Sullam, there is $\alpha < \theta$ such that

$$F_0 := \{ i < \lambda \mid f_\alpha(i) \in V_i \} \in D.$$

This means that for each $i \in F_0$, $g_i(f_\alpha(i)) \subseteq X \cap \lambda_i$ or $g_i(f_\alpha(i)) \subseteq \lambda_i \setminus X$. Let us define a variable c_i which in the first case above $c_i = 0$ and in the second $c_i = 1$. Since D is an ultrafilter, there is a unique $c^* \in \{0,1\}$ such that

$$F_1 := \{i \in F_0 \mid c_i = c^*\} \in D.$$

Suppose without loss of generality that $c^* = 0$. We have that for $i \in F_1$, $g_i(f_\alpha(i)) \subseteq X \cap \lambda_i$. This implies that $X \in U$. Finally to see that U is uniform, if $X \in U$, by definition, $I = \{i < \lambda \mid X \cap \lambda_i \in D_i^+\} \in D$. Since each D_i is uniform, for $i \in I$, $|X \cap \lambda_i| = \lambda_i$ and since D is uniform $|X| = |\bigcup_{i \in I} X \cap \lambda_i| = \sup_{i \in I} \lambda_i = \mu$. This completes the claim. □

To finish the proof, let us construct a base of size at most θ for the ultrafilter U. Let

(1) $\langle d_\alpha \mid \alpha < \theta \rangle$ be a base for D (Assumption (1) of the theorem).
(2) $\langle f_\beta \mid \beta < \theta \rangle$ be the Sullam (Assumption (2) of the theorem).

Define for every $\alpha, \beta < \theta$, the set:

$$B_{\alpha,\beta} = \bigcup_{i \in d_\alpha} g_i(f_\beta(i)).$$

Clearly, each $B_{\alpha,\beta}$ is in U. We now check that $\mathcal{B} = \{B_{\alpha,\beta} \mid \alpha, \beta < \theta\} \subseteq U$ is a base for U. Let $X \in U$, then by the definition, there is $\beta < \theta$ such that

$$H_0 := \{i < \lambda \mid g_i(f_\beta(i)) \subseteq_{D_i} X \cap \lambda_i\} \in D.$$

This implies that for each $i \in H_0$, there is a set $B_i \in D_i$ such that $g_i(f_\beta(i)) \cap B_i \subseteq X \cap \lambda_i$. For every $i < \lambda$, define

$$V_i := \{q \in W_i \mid g_i(q) \subseteq B_i \vee g_i(q) \subseteq \lambda_i \setminus B_i\}.$$

This is open dense, hence by the definition of Sullam, there is $\beta' > \beta$ such that

$$H_1 = \{i < \lambda \mid f_{\beta'}(i) \in V_i\} \in D.$$

Note that if $i \in H_1$, then since $B_i \in D_i$, we have $g_i(f_{\beta'}(i)) \subseteq B_i$. Since $\beta < \beta'$, then by the definition of Sullam,

$$H_2 := \{i < \lambda \mid f_\beta(i) <_{W_i} f_{\beta'}(i)\} \in D.$$

Find $\alpha < \theta$ such that $d_\alpha \subseteq^* H_0 \cap H_1 \cap H_2$, and let $\zeta < \lambda$ be such that $d_\alpha \setminus \zeta \subseteq H_0 \cap H_1 \cap H_2$. To see that $B_{\alpha,\beta'} \setminus \lambda_\zeta \subseteq X$, note that if $\nu \in B_{\alpha,\beta'} \setminus \lambda_\zeta$, then by the definition of $B_{\alpha,\beta'}$, there is $i \in d_\alpha \setminus \zeta$ such that $\nu \in g_i(f_{\beta'}(i))$. Thus, $i \in H_0 \cap H_1 \cap H_2$ and therefore,

$$g_i(f_{\beta'}(i)) \subseteq g_i(f_\beta(i)) \cap B_i \subseteq X \cap \lambda_i.$$

This concludes that $\nu \in X$. □

2.1. **A model where $\mathfrak{u}_{\aleph_{\omega_1}} < 2^{\aleph_{\omega_1}}$.** Now let us turn to force the assumptions of Theorem 2.4 for $\mu = \aleph_{\omega_1}$ with $\theta = \aleph_{\omega_1+1}$ with $2^{\aleph_{\omega_1}} > \aleph_{\omega_1+1}$. Our forcing will be the one from [14], which requires the following assumptions in the ground model V:

- GCH.
- a sequence $\langle \kappa_i \mid i < \omega_1 \rangle$ of strong cardinals with limit κ.
- For each κ_i, E_i is a (κ_i, κ^{++})-extender such that $j_{E_i} : V \to M_{E_i}$ is the extender ultrapower, M_{E_i} computes cardinals correctly up to and including κ^{++}, $M_{E_i}^{\kappa_i} \subseteq M_{E_i}$.
- For each i, we have $s_i : \kappa_i \to \kappa_i$ the function representing κ in j_{E_i}, namely $j_{E_i}(s_i)(\kappa_i) = \kappa$. We can assume that $s_i(\nu) > \max\{\nu, \bar{\kappa}_i\}$ for every ν (see Notation 2.6).
- For each $i_1 < i_2 < \omega_1$, $j_{E_{i_2}}$, there is a function $t^{i_1}_{i_2} : \kappa_{i_2} \to V_{\kappa_{i_2}}$ such that $j_{E_{i_2}}(t^{i_1}_{i_2})(\kappa_{i_2}) = E_{i_1}$ so that $E_{i_1} \in M_{E_{i_2}}$.
- $\square_\kappa$ holds

The last requirement about the square will help us build a Sullam in Theorem 2.17. The assumption can be made possible by working in some canonical model for a Woodin cardinal.

Notation 2.6. for every $\beta \leq \omega_1$ denote by $\bar{\kappa}_\beta = \sup_{\alpha<\beta} \kappa_\alpha$ and $\bar{\kappa}_0 = \omega$. In particular if β is successor then $\bar{\kappa}_\beta = \kappa_{\beta-1}$ and if β is limit then $\bar{\kappa}_\beta < \kappa_\beta$. In particular, $\kappa = \bar{\kappa}_{\omega_1}$

For the convenience of the reader, we include here Merimovich notations for which we will use:

- For $i < \omega_1$, an <u>i-domain</u> is a set $d \in [\kappa^{++}]^{\kappa_i}$ such that $\kappa_i + 1 \subseteq d$ (a set which can be the domain of the Cohen part of a condition in the extender-based forcing).
- Define <u>$mc_i(d)$</u>$=(j_{E_i} \restriction d)^{-1} = \{\langle j_{E_i}(x), x\rangle \mid x \in d\}$. (This is the generator of a measure used by Merimovich in his version of Extender-based forcings).
- Denote the measure generated by $mc_i(d)$, by <u>$E_i(d)$</u>, namely $X \in E_i(d) \iff mc_i(d) \in j_{E_i}(X)$.

A typical element in a measure one set of $E_i(d)$ is a sequence which provide a "layer" of points for the continuation of the Prikry sequences appearing in the domain of a given condition. The following definition summarizes the properties we need from such sequences:

Definition 2.7. An <u>(i,d)-object</u> is a function μ such that:

(1) $\kappa_i \in \mathrm{dom}(\mu) \subseteq d$ and $\mathrm{rng}(\mu) \subseteq s_i(\mu(\kappa_i))^{++} \subseteq \kappa_i$.
 (Since $\mathrm{dom}(mc_i(d)) = j''_{E_i} d$, then $j_{E_i}(\kappa_i) \in \mathrm{dom}(mc_i(d)) \subseteq j_{E_i}(d)$. Also $\mathrm{rng}(mc_i(d)) = d \subseteq \kappa^{++} \subseteq= j_{E_i}(s_i)(\kappa_i))$.
(2) $|\mathrm{dom}(\mu)| = \mu(\kappa_i) < \kappa_i$ and $\mu(\kappa_i)$ is inaccessible.
 (since $|\mathrm{dom}(mc_i(d))| = |d| = \kappa_i < j_{E_i}(\kappa_i)$).

(3) $\mathrm{dom}(\mu) \cap \kappa_i = \mu(\kappa_i)$ and $\mu \restriction \mu(\kappa_i) = id$.
(Since $\kappa_i \subseteq d$, then $\mathrm{dom}(mc_i(d)) \cap j_{E_i}(\kappa_i) = j''_{E_i} d \cap j_{E_i}(\kappa_i) = \kappa_i$. For the second part, note that $\alpha < \kappa_i$, $j_{E_i}(\alpha) = \alpha$ and therefore $mc_i(d)(\alpha) = \alpha$.)

(4) μ is order preserving.
(Since j_{E_i} is order-preserving.)

The set $\underline{OB_i(d)}$ is the set of (i,d)-objects, and clearly $OB_i(d) \in E_i(d)$.

We can omit the 'i' from the "(i,d)-object" and form $OB_i(d)$ since i is uniquely determined by d (recall that $|d| = \kappa_i$).

Definition 2.8. If $d \subseteq d'$ are i-domains let $\pi_{d',d} : OB(d') \to OB(d)$ be the restriction function $\pi_{d',d}(\mu) = \mu \restriction d$ (which is equal to $\mu \restriction \mathrm{dom}(\mu) \cap d$).

Clearly, the generators and the measures are projected using the restriction map, namely $j_{E_i}(\pi_{d',d})(mc_i(d')) = mc_i(d)$ and $(\pi_{d',d})_*(E_i(d')) = E_i(d)$ where $(\pi_{d',d})_*$ is the natural map induced by $\pi_{d',d}$.

Here are two relevant combinatorial lemmas regarding such measures:

Proposition 2.9. *Let $0 \leq i_1 < i_2 < ... < i_n < \omega_1$ and $F : \prod_{k=0}^n A_{i_k} \to X$ is any function such that d_{i_k} is i_k-domain, $A_{i_k} \in E_{i_k}(d_{i_k})$ and $|X| < \kappa_{i_1}$. Then there is $B_{i_k} \subseteq A_{i_k}$ such that $B_{i_k} \in E_{i_k}(d_{i_k})$ such that $F \restriction \prod_{k=0}^n B_{i_k}$ is constant.*

Proposition 2.10. *(The bound for the number of objects with the same projection to the normal measure) For each $i < \omega_1$ and an i-domain d, there is a set $A_i(d)$ such that $A_i(d) \in E_i(d)$, and for each $\nu < \kappa_i$, the size of $\{\mu \in A_i(d) \mid \mu(\kappa_i) = \nu\}$ is at most $s_i(\nu)^{++}$.*

We keep the notation of $A_i(d)$. Finally, we denote the normal measure below E_i by $\underline{E_i(\kappa_i)}$ which is the set of all $X \subseteq \kappa_i$ such that $\kappa_i \in j_{E_i}(X)$. If $A \in E_i(d)$, the projection to normal is denoted by $\underline{A(\kappa_i)}$ and is define as $A(\kappa_i) = \{\mu(\kappa_i) \mid \mu \in A\} \in E_i(\kappa_i)$.

Definition 2.11. A condition in $\mathbb{P}_{\vec{E}}$ is a sequence $p = \langle p_i \mid i < \omega_1 \rangle$ such that there is a finite set $\mathrm{Supp}(p) \in [\omega_1]^{<\omega}$, and we have that:

$$p_i = \begin{cases} \langle f_i, h_i^0, h_i^1, h_i^2 \rangle & i \in \mathrm{Supp}(p) \\ \langle f_i, A_i, H_i^0, H_i^1, H_i^2 \rangle & i \notin \mathrm{Supp}(p) \end{cases}$$

Such that for every every $i_1 < i_2 < \omega_1$, $\mathrm{dom}(f_{i_1}) \subseteq \mathrm{dom}(f_{i_2})$. Denote $\mathrm{Supp}(p) = \{i_1 < i_2 < ... < i_r\}$ and $i_0 = 0$, then for every $i < \omega_1$:

$$\bar{\kappa}_i < \bar{\kappa}_i^{+2} < f_i(\kappa_i) < s_i(f_i(\kappa_i)) < s_i(f_i(\kappa_i))^+ < s_i(f_i(\kappa_i))^{++} < \kappa_i,$$

$f_i(\kappa_i)$ is inaccessible, and we require that:

(1) If there is $k < r$ such that $i \in [i_k, i_{k+1})$ f_i is a partial function from $s_{i_{k+1}}(f_{i_{k+1}}(\kappa_{i_{k+1}}))^{++}$ to κ_i such that $\kappa_i + 1 \subseteq \mathrm{dom}(f_i)$ and $|f_i| = \kappa_i$.

(2) If $i \in [i_r, \omega_1)$, then f_i is a partial function from κ^{++} to κ_i such that $\mathrm{dom}(f_i)$ is an i-domain. In this case, abusively write the forcing in which f_i lives as $Add(\kappa_i^+, \kappa^{++})$.

(3) for $i \in \mathrm{Supp}(p)$, $h_i^0 \in Col(\bar{\kappa}_i^+, < f_i(\kappa_i))$, $h_i^1 \in Col(f_i(\kappa_i), s_i(f_i(\kappa_i))^+)$, $h_i^2 \in Col(s_i(f_i(\kappa_i))^{+3}, < \kappa_i)$.

(4) For $i \notin \mathrm{Supp}(p)$:
  (a) $A_i \in E_i(\mathrm{dom}(f_i))$.
  (b) $\mathrm{dom}(H_i^0) = \mathrm{dom}(H_i^1) = A_i$ and $\mathrm{dom}(H_i^2) = A_i(\kappa_i)$.
  (c) $H_i^0(\mu) \in Col(\bar{\kappa}_i^+, < \mu(\kappa_i))$, $H_i^1(\mu) \in Col(\mu(\kappa_i), s_i(\mu(\kappa_i))^+)$ and $H_i^2(\mu(\kappa_i)) \in Col(s_i(\mu(\kappa_i))^{+3}, < \kappa_i)$.

If p is a condition, we usually represent each component of p by putting the superscript p to that component. For example, f_i in p is denoted by f_i^p. We also write $\mathrm{dom}(f_i^p)$ as d_i^p.

Definition 2.12. The direct order is defined by $p \leq^* q$ if $\mathrm{Supp}(p) = \mathrm{Supp}(q)$, for every i, $f_i^p \subseteq f_i^q$ and

(1) if $i \in \mathrm{Supp}(p)$, then for $h_i^{r,p} \leq h_i^{r,q}$ for $r = 0, 1, 2$.

(2) if $i \notin \mathrm{Supp}(p)$, $\pi_{\mathrm{dom}(f_i^q),\mathrm{dom}(f_i^p)}[A_i^q] \subseteq A_i^p$. $H_i^{r,p}(\pi_{\mathrm{dom}(f_i^q),\mathrm{dom}(f_i^p)}(\mu)) \leq H_i^{r,q}(\mu)$ for every μ and $r = 0, 1$, and $H_i^{2,p}(\gamma) \leq H_i^{r,q}(\gamma)$ for every γ.

Remark 2.13. The collection of μ which is addable to p is of measure-one.

Definition 2.14. Let $i \notin \mathrm{Supp}(p)$. $\mu \in A_i^p$ is *addable* to p if:

(1) $\bar{\kappa}_i < \mu(\kappa_i)$ is inaccessible.

(2) $\cup_{\alpha<i} \mathrm{dom}(f_\alpha) \subseteq \mathrm{dom}(\mu)$ and $\mu \restriction \bar{\kappa}_i = id$.

(3) For every $\beta \in (\max(\mathrm{Supp}(p) \cap \alpha), \alpha)$, $\{\nu \circ \mu^{-1} \mid \nu \in A_\beta^p\} \in t_\alpha^\beta(\mu(\kappa_\alpha))(\mu[\mathrm{dom}(f_\beta)])$.

Definition 2.15. Let $i \notin \mathrm{Supp}(p)$, $i_* = \max(\mathrm{Supp}(p) \cap i)$ where $\max(\emptyset) = -1$, and $\mu \in A_i^p$, define $p + \mu$ as the condition q such that $\mathrm{Supp}(q) = \mathrm{Supp}(p) \cup \{i\}$, and

(1) For $r \in [0, i_*) \cup (i, \omega_1)$, $p_r = q_r$.

(2) For $r = i$, $f_i^q = f_i^p \oplus \mu$, $h_i^{0,q} = H_i^{0,p}(\mu)$, $h_i^{1,q} = H_i^{1,p}(\mu)$ and $h_i^{2,q} = H_i^{2,p}(\mu(\kappa_i))$.

(3) For $r \in [i_*, i)$, $j \geq 0$, $f_r^q = f_r^p \circ \mu^{-1}$ and if $r > i_*$, then $A_r^q = A_r^p \circ \mu^{-1}$, $H_r^{j,q}(\nu) = H_r^{j,q}(\nu \circ \mu)$ for $j = 0, 1$, and $H_j^{2,q} = H_j^{2,p}$.

Define $p + \langle \mu_1, \cdots, \mu_n \rangle$ recursively by $p + (\langle \mu_1, \cdots, \mu_{n-1} \rangle) + \mu_n$. Define an ordering in $\mathbb{P}_{\vec{E}}$ by $p \leq q$ if $p + \vec{\mu} \leq^* q$ for some $\vec{\mu}$ ($\vec{\mu}$ could be empty). Sometimes, we interact an object with the part of the condition that appears before the occurrence of the object, for example, we have a part $p \in \mathbb{P}_{\vec{E}}$, $d \supseteq d_i^p$, and $\mu \in OB_i(d)$, then $p \restriction i$ is considered as an element in $\mathbb{P}_{\vec{E} \restriction i}$, and if $t \in \mathbb{P}_{\vec{E} \restriction i}$, we denote t_μ a tuple obtained by "squishing t by μ, namely we operate as in Definition 2.15 (1) for $r < i_*$ and (3). Note that $t_\mu \in \mathbb{P}_{\langle t_i^\beta(\mu(\kappa_i)) \mid \beta < i \rangle}$.

Proposition 2.16 (Properties). (1) $\mathbb{P}_{\vec{E}}$ *is* κ^{++}*-c.c.*

(2) *For every p, and for every $i \in \mathrm{Supp}(p)$ the forcing above p can be factored to a product*

$$\mathbb{P}_{<i} \times Col(s_i(f_i(\kappa_i)^{+3}, <\kappa_i) \times \mathbb{P}_{>i}$$

Where $(\mathbb{P}_{>i}, \leq^*)$ is a κ_i^+-closed forcing and $|\mathbb{P}_{<i}| \leq s_i(f_i(\kappa_i))^{+2} < \kappa_\alpha$

(3) *$\mathbb{P}_{\vec{E}}$ has the Prikry property and the strong Prikry property (the strong Prikry property says that for every p and a dense open set D, there is $p^* \geq^* p$ and $a \in [\omega_1]^{<\omega}$ such that for every $\vec{\mu} \in \prod_{i\in a} A_i^{p^*}$, $p^* + \vec{\mu} \in D$).*

(4) *Cardinal's structure: In the extension, the κ_i's are preserved and between κ_i and κ_{i+1} we preserve only*

$$\kappa_i^+ < f_{i+1}(\kappa_{i+1}) < s_{i+1}(f_{i+1}(\kappa_{i+1}))^{++} < s_{i+1}(f_{i+1}(\kappa_{i+1}))^{+++}$$

In particular every $i \leq \omega_1$, $\bar{\kappa}_i$ is preserved. κ^+ is preserved by the strong Prikry property, and above κ^{++} we use the chain condition.

(5) *If $\alpha < \omega_1$ is limit, in the extension, $\bar{\kappa}_\alpha = \aleph_\alpha$, $2^{\aleph_\alpha} = \aleph_{\alpha+3}$, κ becomes $\aleph_{\omega_1}$ and $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+2}$. (The mismatch for the cardinals of the powersets of singular cardinals is not a typo. In Section 3 we will elaborate a slight modification so that in the extension, the cardinal behavior on singular cardinals will align uniformly).*

(6) *$\square_{\aleph_{\omega_1}}$ holds. (This is simply because we assume $\square_\kappa$ in the ground model, κ and κ^+ are preserved in the extension, and κ becomes $\aleph_{\omega_1}$).*

Theorem 2.17. *After forcing with $\mathbb{P}_{\vec{E}}$, there is a nice system satisfying the assumption of Theorem 2.4 with $\theta = \aleph_{\omega_1+1}$.*

Proof. The proof is divided into two stages. The first stage is to build a nice system. The second stage is to find a uniform ultrafilter of small base and a Sullam.

Stage 1: We fix any uniform ultrafilter D over ω_1 in the extension. Let us use the sequence $\lambda_i = \kappa_i$ which is regular in the extension. Note that κ_i was measurable in V, we fix a normal measure D'_i on κ_i in V. Since the upper forcing $\mathbb{P}_{>i}$ does not add subsets to κ_i, κ_i remains measurable after forcing with $\mathbb{P}_{>i}$ with the measure D'_i. Also, by the small cardinality of $\mathbb{P}_{<i}$, we can lift any ultrapower embedding using a normal ultrafilter over κ_i from $V^{\mathbb{P}_{>i}}$ to $V^{\mathbb{P}_{>i}\times\mathbb{P}_{<i}}$. Hence κ_i remains measurable after forcing with $\mathbb{P}_{>i} \times \mathbb{P}_{<i}$. The embedding generates a normal measure extending D'_i, and we still call the measure in the extension D'_i. Clearly, the measurability fo κ_i is destroyed by forcing $Col(s_i(f_i(\kappa_i))^{+3}, <\kappa_i)$. However, if $D'_i \in V^{\mathbb{P}_{>i}\times\mathbb{P}_{<i}} =: V_1$ is a normal ultrafilter over κ_i, we can follow the construction in [4, Section 17.1]: Let $j_{D'_i} : V_1 \to M_1$ be the usual ultrapower embedding. Then, $j_{D'_i}(Col(s_i(f_i(\kappa_i)))^{+3}, <\kappa_i))$ is forcing equivalent to $Col(s_i(f_i(\kappa_i)))^{+3}, <\kappa_i) \times \mathbb{Q}$, where $\mathbb{Q}$ adds a collapsing function for every $\alpha \in [\kappa_i, j_{D'_i}(\kappa_i))$ to have cardinality $s_i(f_i(\kappa_i)))^{+3}$. We call the forcing $Col(s_i(f_i(\kappa_i)))^{+3}, I_i)$ where $I_i = [\kappa_i, j_{D'_i}(\kappa_i))$. Over $V_1[G]$ (which is the generic extension by $\mathbb{P}_{\vec{E}}$), let H be $\mathbb{Q}$-generic over $V_1[G]$. and in the model

$V_1[G][H]$ we can lift $j_{D'_i} \subseteq j^* : V_1[G] \to M_1[G * H]$. Now in $V_1[G]$, we define an extension of D'_i:

$$D_i := \{X \subseteq \kappa_i \mid 0_{\mathbb{Q}} \Vdash_{\mathbb{Q}} \kappa_i \in j^*(X)\}$$

Clearly, D_i is uniform. Moreover, in $V_1[G]$ the forcing $(D_i^+, \supseteq)$ is isomorphic to $ro(\mathbb{Q})$[5]. In particular, since $\mathbb{Q}$ is a dense subset of $ro(\mathbb{Q})$, there is a dense embedding $g_i : \mathbb{Q} \to D_i^+$. Let $W_i = \mathbb{Q}$, then $g_i : W_i \to D_i^+$ is order preserving. To see that it is deciding, let $A \subseteq \kappa_i$ and let $q \in \mathbb{Q}$. Then either $A \cap g_i(q) \in D_i^+$ or $(\kappa_i \setminus A) \cap g_i(q) \in D_i^+$. Suppose without loss of generality that $A \cap \pi(q) \in D_i^+$. Then by density of $\mathbb{Q}$, there is $q' \geq q$ such that $g_i(q') \subseteq A \cap g_i(q) \subseteq A$. So far we have proven that $\langle \kappa_i \mid i < \omega_1 \rangle$, $\langle W_i \mid i < \omega_1 \rangle$, $\langle D_i \mid i < \omega_1 \rangle$ and $\langle g_i \mid i < \omega_1 \rangle$ forms a nice system. Note that $\mathbb{Q}$ is the collapse forcing in the sense of V, and $\dot{W}_i$ is decided from any p with $i \in \text{Supp}(p)$, i.e. $\dot{W}_i = Col(s_i(\dot{f}_i(\kappa_i))^{+3}, I_i)$.

Stage 2: We now prove requirements $(1)-(2)$ of Theorem 2.4. (1) is easy, since $\aleph_{\omega_1}$ is singular strong limit, and so $2^{\aleph_1} < \aleph_{\omega_1}$, then $Ch(D) \leq |D| < \aleph_{\omega_1+1}$. For (2) we need the following claim:

Claim 2.18. *For every $p \in \mathbb{P}$ and every sequence $\langle \dot{U}_i \mid i < \omega_1 \rangle$ such that $p \Vdash \dot{U}_i \subseteq \dot{W}_i$ and $\dot{U}_i$ is open dense, there is $p \leq^* p^*$ and a function $F : V_\kappa \to V_\kappa$ in V such that for every generic G of $\mathbb{P}$, there is a translation $F_i^G \in V[G]$ such that $F_i^G \subseteq \dot{W}_i[G]$ is open dense and is a subset of $\dot{U}_i[G]$.*

Proof. Fix p and $\dot{U}_i$ for $i < \omega_1$. Assume for simplicitiy that p is pure. Build a $\leq^*$-increasing sequence $\langle p^i \mid i < \omega_1 \rangle$ such that for each i, $p^{i+1} \restriction (i+1) = p^i \restriction (i+1)$, and at each limit α,we take p^α as a $\leq^*$-least upper bound of $\langle p^\beta \mid \beta < \alpha \rangle$. Let $p^0 = p$. It remains to elaborate the construction at the successor stages. Let $i < \omega_1$ and p^i is constructed. Write $p^i_{i+1} = \langle f, A, H^0, H^1, H^2 \rangle$. Let

$$\mathbb{R}^* = \{(g, \vec{r}) \in Add(\kappa_{i+1}^+, \kappa^{++}) \times \mathbb{P}_{\vec{E} \setminus (i+2)}, \leq^*) \mid$$
$$\text{dom}(g) \text{ is a subset of the domains in the Cohen part of } \vec{r}\}.$$

Let $N \prec H_\rho$ where ρ is a sufficiently large regular cardinal, $p^i, \dot{W}_i, \dot{U}_i, \mathbb{P} \in N$, $\kappa_{i+1} + 1 \subseteq N$, and ${}^{<\kappa_{i+1}}N \subseteq N$. Build an $\mathbb{R}^*$-increasing sequence $\{(f_\gamma, \vec{r}_\gamma) \mid \gamma < \kappa_{i+1}\}$ above $(f, p^i \setminus (i+2))$ such that every initial segment in in N, and for every $\mathbb{R}^*$-dense open set $D \in N$, there is γ such that $(f_\gamma, \vec{r}_\gamma) \in D$. Let $f^* = \cup_\gamma f_\gamma$ and $\vec{r}$ be the $\leq^*$-least upper bound of $\langle \vec{r}_\gamma \mid \gamma < \kappa_{i+1} \rangle$. Then $d^* := \text{dom}(f^*) = N \cap \kappa^{++}$. Let $A^* \in E_i(d^*)$, $A^* \subseteq A_i(d^*)$, and A^* projects down to a subset of A. Then $A^* \subseteq N$. Fix $\gamma < \kappa_{i+1}$. In N, fix $\gamma < \kappa_{i+1}$. for each $\mu \in A^*$ with $\mu(\kappa_{i+1}) = \gamma$, consider $q^\mu = \langle (p^i \restriction (i+1))_\mu, (H^0)(\mu \restriction d_i^{p^i}), (H^1)(\mu \restriction d_i^{p^i}) \rangle$. Define $D_{\mu,x} = \{(h, g, \vec{r}) \in Col(s_i(\gamma)^{+3}, < \kappa_i) \times \mathbb{R}^* \mid \{(t, h^0, h^1) \geq q^\mu \mid$ Either

(1) $t^\frown \langle g \oplus \mu, h^0, h^1, h \rangle^\frown \vec{r} \Vdash x \notin \dot{W}_i$,

[5] Where $ro(\mathbb{Q})$ is the complete boolean algebra of regular open cuts.

(2) or $t^\frown\langle g\oplus\mu, h^0, h^1, h\rangle^\frown\vec{r}\Vdash x\in\dot{W}_i$ and the condition decides some $y\geq x$, $y\in\dot{U}_i\}$,

is open dense. Let $D'_\gamma=\{(g,\vec{r})\in\mathbb{R}^*\mid\exists h(h,g,\vec{r})$ satisfies the strong Prikry property for every $D_{\mu,x}$ with $\mu(\kappa_{i+1})=\gamma\}$. Then $D_{\mu,x}$ is open dense and is in N. The closure of the forcing for $D_{\mu,x}$ is $s_{i+1}(\gamma)^{+3}$. Since the number of such μ is $s_{i+1}(\gamma)^{++}$ and a number of such x is κ_i^+, we have that D'_γ is an open dense set in N. By genericity, $(f^*,\vec{r}^*)\in D'_\gamma$ with a witness $h=:(H^2)^*(\gamma)$. Let $p^{i+1}=p^i\restriction(i+1)^\frown\langle f^*,A^*,(H^0)',(H^1)',(H^2)^*\rangle^\frown\vec{r}^*$ where for $l=0,1$, $(H^l)'$ is the natural map induced from H^l. For the rest of the proof, we denote $d_{i+1}:=\mathrm{dom}(f^*)$ as above.

Take p^* as the $\leq^*$-least upper bound for p^i. For each (μ,x) with $\mu\in A^{p^*}_{i+1}$, by the property of $D_{\mu\restriction d_{i+1},x}$ and the property of p^{i+1}, we have that for each $(\mu\restriction d_{i+1},x)$, there is a set $a_{\mu,x}:=a_{\mu\restriction d_{i+1},x}\in[\omega_1]^{<\omega}$ witnessing the strong Prikry property for p^*, namely for every $\vec{\tau}\in\prod_{\beta\in a_{\mu,x}}A^{p^*}_\beta$, $((H^2)^*(\mu(\kappa_{i+1})),f^{p^*}_{i+1},(p^*+\langle\mu,\vec{\tau}\rangle)\setminus(i+2)\rangle\in D_{\mu\restriction d_{i+1},x}$. For each $\vec{\tau}\in\prod_{i\in a_{\mu,x}}$ maximal, fix a maximal antichain $B_{\mu,x,\vec{\tau}}\subseteq\mathbb{P}_{\langle t^\beta_{i+1}(\mu(\kappa_{i+1}))\mid\beta<i+1\rangle}\times Col(\kappa_i^+,<\mu(\kappa_{i+1}))\times Col(\mu(\kappa_{i+1}),s_{i+1}(\mu(\kappa_{i+1}))^+)$ such that every element in $B_{\mu,x,\vec{\tau}}$ either satisfies (1) or (2). Define $F:V_\kappa\to V_\kappa$ as follows: for each $\mu,x,\vec{\tau}$ and $r\in B_{\mu,x,\vec{\tau}}$, define $F(\mu,x,\vec{\tau},r)=y$ if (2) holds with the decision y. Otherwise, the value is 0. For other elements in $\mathrm{dom}(F)$. assign them as 0.

We now interpret $F[G]$ when G is $\mathbb{P}_{\vec{E}}$-generic containing p^*. For each $x\in W_i:=\dot{W}_i[G]$, find a condition $s\in G$ above $p^*+\langle\mu,\vec{\tau}\rangle$ with $(s\restriction(i+1),(h^0_{i+1})^s,(h^1_{i+1})^s)\geq r$ for a unique $r\in B_{\mu,x,\vec{\tau}}$. Let $y_x=F(\mu,x,\vec{\tau},r)$. Define $F_i^G=\{y\mid y\geq y_x$ for some $x\in W_i\}$. We see that F_i^G is open dense and is a subset of $\dot{U}_i[G]$. □

From the claim we get that if $\langle U_i\mid i<\omega_1\rangle\in V[G]$ is a list of dense open subsets of $\langle W_i\mid i<\omega_1\rangle$, then there is a function $F:V_\kappa\to V_\kappa\in V$ such that for every $i<\omega_1$, $F_i^G\subseteq U_i$ and F_i^G is dense open. Since we have GCH in V, we can enumerate $\langle g_\alpha\mid\alpha<\kappa^+\rangle$ all the functions $F:V_\kappa\to V_\kappa$ and in $V[G]$, denote $U_{\alpha,i}=(g_\alpha)_i^G$ and $\vec{U}_\alpha=\langle U_{\alpha,i}\mid i<\omega_1\rangle$. Then the sequence $\langle\vec{U}_\alpha\mid\alpha<\kappa^+\rangle$ has the following properties:

(1) Each $\vec{U}_\alpha$ is a sequence of dense open subsets $U_{\alpha,i}\subseteq W_i$.
(2) For every sequence $\langle V_i\mid i<\omega_1\rangle$ of dense open subsets of W_i, there is some $\alpha<\kappa^+$ such that for every $i<\omega_1$, $U_{\alpha,i}\subseteq V_i$.

Let us now define a Sullam $\langle f_\alpha\mid\alpha<\aleph_{\omega_1+1}\rangle$ modulo the filter of co-bounded subsets of ω_1 in the generic extension. Fix a $\square_{\aleph_{\omega_1}}$-sequence $\langle C_\alpha\mid\alpha\in lim(\aleph_{\omega_1+1})\rangle$ such that each C_α has order-type below $\aleph_{\omega_1}$. Our induction hypothesis is that for each limit α, if i^* is the least such that the closure of W_{i^*} is strictly greater than $ot(\lim(C_\alpha))$, the for $i\geq i^*$, $\langle f_\beta(i)\mid\beta\in\lim(C_\alpha)\cup\{\alpha\}\rangle$ is strictly increasing. f_0 is random. Fix f_α, let $f_{\alpha+1}$ be such that for all i, $f_\alpha(i)<_{W_i}f_{\alpha+1}(i)$ and $f_{\alpha+1}(i)\in U_{\alpha,i}$. Now, assume α is limit.

If $ot(C_\alpha) = \omega$, then let $f_\alpha(i) = \sup_{\beta \in C_\alpha} f_\beta(i)$. A straightforward argument shows that f_α is a $\leq^*$-upper bound of $\langle f_\beta \mid \beta < \alpha\rangle$. Assume that $ot(C_\alpha) > \omega$. Let i^* be the least such that the closure of W_{i^*} is greater than $ot(\lim(C_\alpha))$. We divide further into two subcases. If $\lim(C_\alpha)$ is bounded in α, then $\beta^* = \max(\lim(C_\alpha))$ exists. This only happens if $\mathrm{cf}(ot(C_\alpha)) = \omega$ and $C_\alpha \setminus (\beta^* + 1)$ has order-type ω. For this case, let $i^{**} \geq i^*$ be such that for $i \geq i^{**}$, $\langle f_\beta(i)\rangle^\frown \langle f_{\gamma(i)} \mid i \in C_\alpha \setminus (\beta + 1)\rangle$ is strictly increasing. Define $f_\alpha(i)$ such that for $i \in [i^*, i^{**})$, $f_\alpha(i) = f_\beta(i)$, and for $i \geq i^{**}$, $f_\alpha(i) = \sup_{\gamma \in C_\alpha \setminus (\beta+1)} f_\alpha(i)$. Then f_α is a $\leq^*$ upper bound of $\langle f_\gamma \mid \gamma < \alpha\rangle$ and f_α satisfies the induction hypothesis. We now consider the second subcase, which is the case where $\lim(C_\alpha)$ is unbounded in α. In this case, for $i \geq i^*$, $\langle f_\beta(i) \mid \beta \in \lim(C_\alpha)\rangle$ is increasing. Let f_α be such that for $i \geq i^*$, $f_\alpha(i) = \sup_{\beta \in \lim(C_\alpha)} f_\beta(i)$. This completes the proof of Theorem 2.17.

□

From Theorem 2.4 and Theorem 2.17, we conclude that

Theorem 2.19. *Assume GCH, $\langle \kappa_\alpha \mid \alpha < \omega_1\rangle$ is an increasing sequence of cardinals such that by letting $\kappa = \sup_{\alpha < \omega_1} \kappa_\alpha$,*

1. *for each α, κ_α carries a $(\kappa_\alpha, \kappa^{++})$-extender E_α.*
2. *let $j_\alpha : V \to Ult(V, E_\alpha)$, then $Ult(V, E_\alpha)$ computes cardinals correctly up to and including κ^{++}.*
3. *if $\beta < \alpha$, there is t^β_α such that $j_\alpha(t^\beta_\alpha)(\kappa_\alpha) = E_\beta$.*

Then, there is a forcing such that in a generic extension, $\mathfrak{u}_{\aleph_{\omega_1}} < 2^{\aleph_{\omega_1}} = \aleph_{\omega_1+2}$.

Remark 2.20. With the same argument, in the forcing extension from Theorem 2.19, for any limit ordinal $\alpha < \omega_1$ we also get $\mathfrak{u}_{\aleph_\alpha} < 2^{\aleph_\alpha}$.

3. Forcing with a long sequence of overlapping extenders with collapses

We start with a model of GCH, and a sequence $\langle \kappa_i \mid i < \kappa\rangle$, $\kappa = \sup_{\alpha<\kappa} \kappa_i$. Assumptions in V:

- GCH.
- κ is inaccessible.
- For each κ_i, E_i is a (κ_i, κ^{++})-extender such that $j_{E_i} : V \to M_{E_i}$ is the extender ultrapower, M_{E_i} computes cardinals correctly up to and including κ^{++}, $M_{E_i}^{\kappa_i} \subseteq M_{E_i}$.
- For each i, we have $s_i : \kappa_i \to \kappa_i$ the function representing κ in j_{E_i}, namely $j_{E_i}(s_i)(\kappa_i) = \kappa$. We can assume that $s_i(\nu) > \max\{\nu, \bar{\kappa}_i\}$ for every ν.
- For each $i_1 < i_2 < \kappa$, there is a function $t^{i_1}_{i_2} : \kappa_{i_2} \to V_{\kappa_{i_2}}$ such that $j_{E_{i_2}}(t^{i_1}_{i_2})(\kappa_{i_2}) = E_{i_1}$, and in particular $E_{i_1} \in Ult(V, E_{i_2})$.
- κ is the least such cardinal.
- For each $\alpha < \kappa$ limit, $\square_{\sup_{\beta<\alpha}\kappa_\beta}$. holds.

Remark 3.1. Note that for such κ, κ is an inaccessible cardinal and for every limit $\alpha < \kappa$, $\sup_{i<\alpha} \kappa_i$ is singular, otherwise, $\sup_{i<\alpha} \kappa_i$ is also inaccessible, and hence $\alpha = \sup_{i<\alpha} \kappa_i$.

Notations:
for every $\beta \leq \kappa$ denote by $\bar{\kappa}_\beta = \sup_{\alpha<\beta} \kappa_\alpha$. In particular if β is successor then $\bar{\kappa}_\beta = \kappa_{\beta-1}$ and if β is limit then $\bar{\kappa}_\beta < \kappa_\beta$. Also, $\kappa = \bar{\kappa}_\kappa$. Also note that for each β, $\beta \leq \bar{\kappa}_\beta < \kappa_\beta$.

Merimovich notations:

- For $i < \kappa$, an <u>i-domain</u> is a set $d \in [\kappa^{++}]^{\kappa_i}$ such that $\kappa_i + 1 \subseteq d$
- Define $\underline{mc_i(d)}$=$(j_{E_i} \restriction d)^{-1} = \{\langle j_{E_i}(x), x\rangle \mid x \in d\}$ (This is the generator of a measure used by Merimovich in his version of Extender-based forcings).
- Denote the measure generated by $mc_i(d)$, by $\underline{E_i(d)}$, namely $X \in E_i(d) \iff mc_i(d) \in j_{E_i}(X)$.

We define a typical element in a measure one set of $E_i(d)$. It is a sequence which will provide a "layer" of points for the continuation of the Prikry sequences appearing in the domain of a given condition. The proof is simply to reflect the properties of the generator $mc_i(d)$.

Definition 3.2. An <u>(i,d)-object</u> is a sequence/function μ such that:

(1) $\kappa_i \in \mathrm{dom}(\mu) \subseteq d$, $\mathrm{rng}(\mu) \subseteq s_i(\mu(\kappa_i))^{++}$.
(2) $|\mathrm{dom}(\mu)| = \mu(\kappa_i) < \kappa_i$ and $\mu(\kappa_i)$ is inaccessible.
(3) $\mathrm{dom}(\mu) \cap \kappa_i = \mu(\kappa_i)$ and $\mu \restriction \mu(\kappa_i) = id$.
(4) μ is order preserving.

The set $\underline{OB_i(d)}$ is the set of (i,d)-objects, and $OB_i(d) \in E_i(d)$ (see the arguments in Definition 2.7). If d is clear from the context, and μ is an (i,d)-object, we denote $i_\mu = i$. If $\vec{\mu} = \langle \mu_1, \cdots, \mu_n\rangle$ is a sequence of objects, where $i_{\mu_1} < \cdots < i_{\mu_n}$, denote $i_{\vec{\mu}}$ the ordinal i_{μ_n}.

We can omit the 'i' from the "(i,d)-object" and from $OB_i(d)$ since i is determined by d (recall that $|d| = \kappa_i$).

The projections: At the price of complicating the notations what we gain is that the projections between the measures of the extender are just restriction:

Definition 3.3. If $d \subseteq d'$ are i-domains, let $\pi_{d',d} : OB(d') \to OB(d)$ be the restriction function $\pi_{d',d}(\mu) = \mu \restriction d$ (which is equal to $\mu \restriction \mathrm{dom}(\mu) \cap d$).

Clearly the generators and the measures are projected using the restriction map:

Proposition 3.4. (1) $j_{E_i}(\pi_{d',d})(mc_i(d')) = mc_i(d)$.
(2) $(\pi_{d',d})_*(E_i(d')) = E_i(d)$, *where* $(\pi_{d',d})_*$ *is the natural induced map from* $\pi_{d',d}$.

Proposition 3.5. *(The bound for the number of objects with the same projection to the normal measure) For each* $i < \kappa$ *and an* i*-domain* d*, there* *is a set $A_i(d)$ such that $A_i(d) \in E_i(d)$, and for each $\nu < \kappa_i$, the size of $\{\mu \in A_i(d) \mid \mu(\kappa_i) = \nu\}$ is at most $s_i(\nu)^{++}$.*

We keep the notation of $A_i(d)$. We also need notations for the normal measure:

- The normal measure $\underline{E_i(\kappa_i)}$ is the set of all $X \subseteq \kappa_i$ such that $\kappa_i \in j_{E_i}(X)$.
- If $A \in E_i(d)$ (then recall that $\kappa_i \in d$) and the projection to normal is denoted by $\underline{A(\kappa_i)}$ and is define as $A(\kappa_i) = \{\mu(\kappa_i) \mid \mu \in A\} \in E_i(\kappa_i)$

Definition 3.6. $\mathbb{P}_{\langle E_i | i<\kappa\rangle}$ is a sequence $p = \langle p_i \mid i < \kappa\rangle$ such that there is a finite set $\mathrm{Supp}(p) \in [\kappa]^{<\omega}$, and we have that:

$$p_i = \begin{cases} \langle f_i, h_i^0, h_i^1\rangle & i \in \mathrm{Supp}(p) \\ \langle f_i, A_i, H_i^0, H_i^1\rangle & i \notin \mathrm{Supp}(p) \end{cases}$$

Such that for every $i_1 < i_2 < \kappa$, $\mathrm{dom}(f_{i_1}) \subseteq \mathrm{dom}(f_{i_2})$. Denote $\mathrm{Supp}(p) = \{i_1 < i_2 < ... < i_r\}$, then for every $i < \kappa$:

$$\bar{\kappa}_i < \bar{\kappa}_i^{+2} < f_i(\kappa_i) < s_i(f_i(\kappa_i)) < s_i(f_i(\kappa_i))^+ < s_i(f_i(\kappa_i))^{++} < \kappa_i,$$

and we require that:

(1) If there is $k < r$ such that $i \in [i_k, i_{k+1})$ (where $i_0 = 0$), f_i is a partial function from $s_{i_{k+1}}(f_{i_{k+1}}(\kappa_{i_{k+1}}))^{++}$ to κ_i such that $\kappa_i + 1 \subseteq \mathrm{dom}(f_i)$ and $|f_i| = \kappa_i$.
(2) If $i \in [i_r, \kappa)$, then f_i is a partial function from κ^{++} to κ_i such that $\mathrm{dom}(f_i)$ is an i-domain. We will abusively write "$f_i \in Add(\kappa^{++}, \kappa_i^+)$".
(3) for $i \in \mathrm{Supp}(p)$, $h_i^0 \in Col(\bar{\kappa}_i^{+2}, s_i(f_i(\kappa_i))^+)$, $h_i^1 \in Col(s_i(f_i(\kappa_i))^{+3}, <\kappa_i)$. (So in the generic extension $V[G]$ we will have: $\bar{\kappa}_i < (\bar{\kappa}_i^+)^{V[G]} = \bar{\kappa}_i^V < (\bar{\kappa}_i^{++})^{V[G]} = (s_i(f_i(\kappa_i))^{++})^V < (\bar{\kappa}_i^{+3})^{V[G]} = (s_i(f_i(\kappa_i))^{+3})^V < (\bar{\kappa}_i^{+4})^{V[G]} = \kappa_i.$).
(4) For $i \notin \mathrm{Supp}(p)$:
  (a) if there is $k < r$ such that $i \in [i_k, i_{k+1})$, then $A_i \in t^i_{i_{k+1}}(f_{i_{k+1}}(\kappa_{i_{k+1}}))(\mathrm{dom}(f_i))$.
  (b) if $i > i_r$, then $A_i \in E_i(\mathrm{dom}(f_i))$.
  (c) $\mathrm{dom}(H_i^0) = A_i$ and $\mathrm{dom}(H_i^1) = A_i(\kappa_i)$.
  (d) $H_i^0(\mu) \in Col(\bar{\kappa}_i^+, s_i(\mu(\kappa_i))^+)$ and $H_i^2(\mu(\kappa_i)) \in Col(s_i(\mu(\kappa_i))^{+3}, <\kappa_i)$.

If p is a condition, we usually represent each component of p by putting the superscript p to that component. For example, f_i in p is denoted by f_i^p. We also write $\mathrm{dom}(f_i^p)$ as d_i^p.

Remark 3.7. The collapses are different from Definition 2.11. There is a flexibility to split collapses to be as in Definition 2.11 or merge some collapses as in Definition 3.6. The reason is, we want to demonstrate a flexibility on the cardinal arithmetic on regular cardinals. Ultimately, we will obtain a ZFC model V_κ where GCH holds at regulars, SCH fails at singulars, small ultrafilter numbers everywhere, and κ is the least strongly inaccessible cardinal.

The direct extension is clear:

Definition 3.8. The direct order is defined by $p \leq^* q$ if $\mathrm{Supp}(p) = \mathrm{Supp}(q)$, for every i, $f_i^p \subseteq f_i^q$ and for

1. If $i \in \mathrm{Supp}(p)$ then for $h_i^{r,p} \leq h_i^{r,q}$ for $r = 0, 1$.
2. If $i \notin \mathrm{Supp}(p)$, $\pi_{\mathrm{dom}(f_i^q),\mathrm{dom}(f_i^p)}[A_i^q] \subseteq A_i^p$. $H_i^{0,p}(x) \leq H_i^{0,q}(\pi_{\mathrm{dom}(f_i^q),\mathrm{dom}(f_i^p)}(x))$ for all x, and $H_i^{1,p}(\gamma) \leq H_i^{1,q}(\gamma)$ for all γ.

Definition 3.9. Let $i \notin \mathrm{Supp}(p)$. $\mu \in A_i^p$ is addable to p if:

1. $\bar{\kappa}_i < \mu(\kappa_i)$ is inaccessible.
2. $\cup_{\alpha < i} \mathrm{dom}(f_\alpha) \subseteq \mathrm{dom}(\mu)$ and $\mu \restriction \bar{\kappa}_i = id$.
3. For every $\beta \in (\max(\mathrm{Supp}(p) \cap i), i)$, $\{\nu \circ \mu^{-1} \mid \nu \in A_\beta^p\} \in t_i^\beta(\mu(\kappa_i))(\mu[\mathrm{dom}(f_\beta)])$.

Remark 3.10. The collection of $\mu \in A_i^p$ which is addable to p is of measure-one, since $i \leq \bar{\kappa}_i < \kappa_i$.

Definition 3.11. Let $i \notin \mathrm{Supp}(p)$, $i_* = \max(\mathrm{Supp}(p) \cap i)$, where $\max(\emptyset) = -1$ and $\mu \in A_i^p$, define $p+\mu$ as the condition q such that $\mathrm{Supp}(q) = \mathrm{Supp}(p) \cup \{i\}$, and

1. For $r \in [0, i_*) \cup (i, \kappa)$, $p_r = q_r$.
2. For $r = i$, $f_i^q = f_i^p + \mu$, $h_i^{0,q} = H_i^{0,p}(\mu)$, and$h_i^{1,q} = H_i^{1,p}(\mu(\kappa_i))$.
3. For $r \in [i_*, i)$ with $r \geq 0$, $f_r^q = f_r^p \circ \mu^{-1}$, if $r > i^*$, then $A_r^q = A_r^p \circ \mu^{-1}$. $H_r^{0,q}(\nu) = H_r^{0,q}(\nu \circ \mu)$ and $H_r^{1,q} = H_r^{1,p}$. Finally, if $i_* \geq 0$, then $h_{i_*}^{0,q} = h_{i_*}^{0,p}$ and $h_{i_*}^{1,q} = h_{i_*}^{1,p}$.

Define $p + \vec{\mu}$ recursively by $p + \langle \mu_1, \cdots, \mu_n \rangle = (p + \langle \mu_1, \cdots, \mu_{n-1} \rangle) + \mu_n$. We define $p \leq q$ if $p + \vec{\mu} \leq^* q$ for some $\vec{\mu}$ ($\vec{\mu}$ could be empty). Sometimes, we interact an object with the part of the condition that appears before the occurrence of the object, for example, we have a part $p \in \mathbb{P}_{\vec{E}}$, $d \supseteq d_i^p$, and $\mu \in OB_i(d)$, then $p \restriction i$ is considered as an element in $\mathbb{P}_{\vec{E} \restriction i}$, and if $t \in \mathbb{P}_{\vec{E} \restriction i}$, we denote t_μ a tuple obtained by "squishing t by μ, namely we operate as in Definition 3.11 (1) for $r < i_*$ and (3). Note that $t_\mu \in \mathbb{P}_{\langle t_i^\beta(\mu(\kappa_i)) \mid \beta < i \rangle}$.

The following definition follows from [1].

Definition 3.12. Let p be a condition, $n > 0$. A (p, n)-fat-tree is a tree T of height n such that the following hold:

1. $Level_k(T)$ is a collection of sequences of objects of length $k + 1$.
2. $Level_0(T) \in E_i(d_i^p)$ for some i.
3. If $\vec{\mu} = \langle \mu_1, \cdots, \mu_k \rangle \in T$ and $k < n-1$, then there is $i > i_{\vec{\mu}}$ (the definition of $i_{\vec{\mu}}$ is as in Definition 3.2) such that $Succ_T(\vec{\mu}) \in E_i(\mathrm{dom}(f_i^p))$.

We say that T is *fully compatible* with p if for every non-maximal $\vec{\mu} \in T$, $Succ_T(\vec{\mu}) = A_{i_{\vec{\mu}}}^p$.

The following two lemmas are easy.

Lemma 3.13. *Let $p \in \mathbb{P}_{\vec{E}}$ and T be a (p,n)-fat-tree.*

1. *If T is fully compatible with p, then the collection $\{p + \vec{\mu} \mid \vec{\mu} \in T$ is maximal$\}$ is predense above p.*
2. *There is $p^* \geq^* p$ and a (p^*,n)-fat tree T^* with T^* is a subtree of T of the same height such that T^* of the same height, and T^* is fully compatible with p^*.*

Lemma 3.14. *Let T be a (p,n)-fat tree and $F : \{\vec{\mu} \in T \mid \vec{\mu}$ is maximal$\} \to \gamma$, $\gamma < \kappa_i$ where $Level_0(T) \in E_i(d)$ for some d. Then there is a fat subtree $T' \subseteq T$ of the same height such that $F \restriction \{\vec{\mu} \in T' \mid \vec{\mu}$ is maximal$\}$ is constant.*

Lemma 3.15 (The integration lemma [14])**.** *Let p be a condition $i \notin \mathrm{Supp}(p)$, $d^* \supseteq d_i^p$, $A^* \restriction d_i^p \subseteq A_i^p$. Suppose that for each $\mu \in A^*$, lett$(\mu) \geq^* (p \restriction i)_\mu$ and $h^0(\mu) \geq (H_i^0)^p(\mu \restriction d_i^p)$. Then there is $p^* \geq^* p$ such that for each $\tau \in A_i^{p^*}$ with $\mu = \tau \restriction d^*$, $(p^* \restriction i)_\tau = t(\mu)$ and $(H_i^0)^{p^*}(\tau) = h^0(\mu)$.*

Theorem 3.16 (The strong Prikry property)**.** *Let D be a dense open subset of $\mathbb{P}_{\vec{E}}$ and p be a condition. Then there is a direct extension $p^* \geq^* p$ and a (p^*,n)-fat-tree T, for some n, which is fully compatible with p^* such that for every maximal $\vec{\mu} \in T$, $p^* + \vec{\mu} \in D$.*

Remark 3.17. The proof of the strong Prikry property requires an induction of the length of the sequence of extenders. The proof where the sequence has short length was shown in [14]. The proofs for the forcings from longer sequences of extenders where the lengths are below κ are essentially the same as the proof of Theorem 3.18. We shall only show the strong Prikry property for $\mathbb{P}_{\vec{E}}$ while we apply the Prikry property of the forcings where the lengths of the sequences of extenders are below κ implicitly.

Proof of Theorem 3.16. Let p be a condition and D be a dense open set. If there is $p^* \geq^* p$ such that $p^* \in D$, then the proof is done. Suppose it is not the case. For simplicity, assume p is pure. The plan is to build $\langle p^n \mid 0 < n < \omega \rangle$ such that for each n, either every direct extension of an n-step extension of p^n is not in D, or there is a (p^n,n)-fat-tree T^n fully compatible with p^n such that every n-step extension of p^n using a maximal node in T is in D.

Stage 1: step A We build a $\leq^*$-increasing sequence $\langle q^i \mid i < \kappa \rangle$ such that

1. $q^0 \geq^* p$.
2. for $i' < i$, $q^i_{i'} = q^{i'}_{i'}$.

In the end, we can take q^* such that $q^*_i = q^i_i$. Then $q^* \geq^* p$. q^* will satisfy Claim 3.18. Fix $i < \kappa$. Assume $q^{i'}$ is constructed for $i' < i$. Let q' be such that for all j, q'_j is the weakest " $\leq^*$ "-upper bound bound of $\{q^{i'}_j \mid i' < i\}$, namely we take the union of Cohen functions, intersect the measure-one sets, and their functions whose outputs are collapses, we take the pointwise-lower bound. Note that for $i' < i$, $q'_{i'} = q^{i'}_{i'}$. Clearly, q' is a $\leq^*$-lower bound of $\{q^{i'} \mid i' < i\}$. Write $q'_i = \langle f, A, H^0, H^1 \rangle$. Define

$$\mathbb{R}_i^* = \{(g,r) \mid g \in Add(\kappa^{++},\kappa_i^+),\ r \in (\mathbb{P}_{\langle E_\beta \mid \beta > i\rangle}, \leq^*), \text{ and } \mathrm{dom}(g) \text{ is a subset of the domains of Cohen parts in } r\}$$

Let $N \prec H_\theta$ for some sufficiently large θ, ${}^{<\kappa_i}N \subseteq N$, $\kappa_i, q', \mathbb{P}, \mathbb{R}_i^*, D \in N$, $|N| = \kappa_i$. Enumerate dense open subsets of $\mathbb{R}_i^*$ in N as $\langle D_\alpha \mid \alpha < \kappa_i\rangle$ such that every proper initial segment is in N. Build an $\mathbb{R}_i^*$-increasing sequence $\langle (f_\alpha, r_\alpha) \mid \alpha < \kappa\rangle$ above $(f, q' \setminus (i+1))$ such that $(f_\alpha, r_\alpha) \in D_\alpha$ for all α. Let $f^* = \cup_{\alpha<\kappa_i} f_\alpha$ and r^* be the minimal $\leq^*$-upper bound of $\langle r_\alpha \mid \alpha < \kappa\rangle$. Then (f^*, r^*) is $(N, \mathbb{R}_i^*)$-generic in a strong sense: for $D' \in N$ open dense subset of $\mathbb{R}_i^*$, there is $(f', r') \in D'$ such that $(f^*, r^*) \geq (f', r') \geq (f, q' \setminus (i+1))$. Note that $d^* := \mathrm{dom}(f^*) = N \cap \kappa^{++}$. Let $A^* \in E_i(d^*)$, $A^* \subseteq A_i(d^*)$ ($A_i(d^*)$ is as in Lemma 3.5), and A^* project down to a subset of A. Then $A^* \subseteq N$. Fix $\gamma \in A^*(\kappa_i)$. In N, let $\{(t_\alpha, \mu_\alpha, h^0_\alpha) \mid \alpha < s_i(\gamma)^{++}\}$ be an enumeration of (t, μ, h^0) such that $t \in \mathbb{P}_{\langle t_i^\beta(\gamma) \mid \beta < i\rangle}$, $\mu \in A^*$ with $\mu(\kappa_i) = \gamma$, $h^0 \in Col(\bar{\kappa}_i^+, s_i(\gamma)^+)$. Let D_γ be the collection $(g, r) \in \mathbb{R}_i^*$ such that for all $\alpha < s_i(\gamma)^{++}$,

- $\mathrm{dom}(\mu_\alpha) \subseteq \mathrm{dom}(g)$.
- there is $h^1 \geq H^1(\gamma)$ such that if there are $g' \geq g \oplus \mu_\alpha$, $h' \geq H^1(\gamma)$, and $r' \geq^* r$ with
$$t_\alpha {}^\frown \langle g', h^0_\alpha, h'\rangle {}^\frown r' \in D,$$
then
$$t_\alpha {}^\frown \langle g \oplus \mu_\alpha, h^0_\alpha, h^1\rangle {}^\frown r \in D.$$

Since $\mathbb{R}_i^*$ and $Col(s_i(\gamma)^{+3}, < \kappa_i)$ are $s_i(\gamma)^{+3}$-closed, $D_\gamma \in N$ is open dense and is in N. By genericity, $(f^*, r^*) \in D_\gamma$ with a witness h^1. Define $(H^1)^*(\gamma) = h^1$. Let q^i be such that $q^i \restriction i = q' \restriction i$, $q^i_i = \langle f^*, A^*, (H^0)^*, (H^1)^*\rangle {}^\frown r^*$, where $(H^0)^*(\tau) = (H^0)(\tau \restriction \mathrm{dom}(f))$. q^i has the following property: for each $\mu \in A_i^{q^i}$, if t, g, h^0, h^1 and r are such that
$$t^\frown \langle g, h^0, h^1\rangle^\frown r \geq q^i + \langle \mu\rangle,\ r \geq^* (q^i + \langle\mu\rangle) \setminus (i+1) (\text{which is } q^i \setminus (i+1)),$$
and
$$t^\frown \langle g, h^0, h^1\rangle^\frown r \in D,$$
then
$$t^\frown \langle f_i^{q^i} \oplus \mu, h^0, (H^1)^{q^i}(\mu(\kappa_i))\rangle^\frown (q^i \setminus (i+1)) \in D.$$
Recall that we take q^* such that $q^*_i = q^i_i$ for all i.

Claim 3.18. *For all $i < \kappa$, $\mu \in A_i^{q^*}$, if there are t, g, h^0, h^1, and r such that*
$$t^\frown \langle g, h^0, h^1\rangle^\frown r \geq q^* + \langle\mu\rangle,\ r \geq^* (q^* + \langle\mu\rangle) \setminus (i+1),$$
and
$$t^\frown \langle g, h^0, h^1\rangle^\frown r \in D,$$
then

$$t^\frown\langle f_i^{q^*}\oplus\mu, h^0, (H_i^1)^{q^*}(\mu(\kappa_i))\rangle^\frown(q^*\setminus(i+1))\in D.$$

Proof. The claim is just a consequence of the property of q^i for all i. □

Step B We now consider the following two cases:

Case 1: For each $i<\kappa$, the collection B_i of $\mu\in A_i^{q^*}$ such that "there are t, h^0 such that $t\geq^*(q^*+\langle\mu\rangle)\restriction i$, $h^0\geq(H_i^0)^{q^*}(\mu)$, and $t^\frown\langle f_i^{q^*}\oplus\mu, h^0, (H_i^1)^{q^*}(\mu(\kappa_i))\rangle^\frown q^*\setminus(i+1)\in D$" is of measure-zero. In this case, let p^1 be such that $p_i^1=\langle f_i^{q^*}, B_i^*, (H_i^0)^{q^*}\restriction B_i^*, (H_i^1)^{q^*}\restriction(B_i^*(\kappa_i))\rangle$, where $B_i^*=A_i^{q^*}\setminus B_i$.

Case 2: The negation of Case 1. This means that there is $i<\kappa$, a measure-one set $B\subseteq A_i^{q^*}$ such that for each $\mu\in B$, there are $t=t(\mu)$ and $h^0=h^0(\mu)$ such that $t\geq^*(q^*+\langle\mu\rangle)\restriction i$ and $h^0\geq(H_i^0)^{q^*}(\mu)$ such that

$$t^\frown\langle f_i^{q^*}\oplus\mu, h^0, (H_i^1)^{q^*}(\mu(\kappa_i))\rangle^\frown(q^*\setminus(i+1))\in D.$$

Let i be the least such. Use Lemma 3.15 with $t(\mu)$ and $h^0(\mu)$ to obtain p^1 such that for all $\tau\in A_i^{p^1}$ with $\mu=\tau\restriction d_i^{q^*}$, $(p^1\restriction i)_\tau=t(\mu)$ and $(H_i^0)^{p^1}(\tau)=h^0(\mu)$.

We now have that $p^1\geq^* p$ and for $\tau\in A_i^{p^1}$, $(p^1\restriction i)_\tau=t(\tau\restriction d_i^{q^*})$. With the property of q^*, one can check that if there is a direct extension of a one-step extension of p^1 entering D, then every one-step extension of p^1 using an object in $A_i^{p^1}$ is in D. If this is the case, let $p^*=p^1$, and then we are done.

Stage $n(1<n<\omega)$:

Remark 3.19. Note that the proof for Stage 1 holds for any condition. Furthermore, by induction hypothesis, we will assume that for any condition r (in any slight variation of the long extender-based forcing with collapses, e.g. $\mathbb{P}_{\vec{E}\setminus i}$ for some i), and a dense open set D^*, for $k<n$, there is $r^*\geq^* r$ such that if there is a direct extension of a k-step extension of r^* entering D^*, then there is a (k,r^*)-fat tree S^* fully compatible with r^* such for every $\vec{\tau}\in S^*$ maximal, $r^*+\vec{\tau}\in D^*$. The statement holds for the exact proof as in Stage 1.

step A This will be similar to step A in Stage 1, except that the dense sets we are considering in this stage are more complicated. Suppose p^k has been constructed for $k<n$. Assume that if there is an $n-1$-step extension of p^{n-1} being in D, then there is $(p^{n-1},n-1)$-fat tree fully compatible with p^{n-1} such that every extension of p^{n-1} using a maximal node in the tree belongs to D. Let $p'=p^{n-1}$.

Build a $\leq^*$-increasing sequence $\langle q^i\mid i<\kappa\rangle$ such that

(1) $p'\leq^* q^0$.
(2) for $i'<i$, $q_{i'}^i=q_{i'}^{i'}$.

Again, we take q^* such that $q^*_i = q^i_i$ for all i and q^* will satisfy a certain property. Fix $i < \kappa$, and assume for $i' < i$, $q_{i'}$ is constructed. Let q' be the least $\geq^*$-upper bound of $\langle q^{i'} \mid i' < i \rangle$. Hence, for $i' < i$, $q'_{i'} = q^{i'}_{i'}$. Write $q'_i = \langle f, A, H^0, H^1 \rangle$, $d = \mathrm{dom}(f)$. Let $\mathbb{R}^*_i$ be as in Stage 1 and

$$\mathbb{R}_i = \{(g, r) \mid g \in Add(\kappa^{++}, \kappa_i^+), r \in (\mathbb{P}_{\langle E_\beta \mid \beta > i \rangle}, \leq), \text{ and } \mathrm{dom}(g) \text{ is a subset of the domains of Cohen parts in } r\}.$$

Note that $\mathbb{R}_i$ and $\mathbb{R}^*_i$, as sets, are equal. The difference is the ordering. Let $N \prec H_\theta$ for some sufficiently large θ, ${}^{<\kappa_i}N \subseteq N$, $\kappa_i, \mathbb{P}, \mathbb{R}^*_i, \mathbb{R}_i, D, q' \in N$, $|N| = \kappa_i$, $\kappa_i + 1 \subseteq N$, and let $\langle (f_\alpha, r_\alpha) \mid \alpha < \kappa_i \rangle$ be an $\mathbb{R}^*_i$-increasing sequence above $(f, q' \setminus (i+1))$ such that every dense set contains an element in the sequence, and every proper initial segment of the sequence is in N. By letting $f^* = \cup_{\alpha < \kappa_i} f_\alpha$, and r^* the least $\leq^*$-upper bound of $\langle r_\alpha \mid \alpha < \kappa \rangle$, then (f^*, r^*) is $(N, \mathbb{R}^*_i)$-generic in the strong sense, as described in Stage 1, Step A. Let $d^* = \mathrm{dom}(f^*) = N \cap \kappa^{++}$, $A^* \in E_i(d^*)$ project down to a subset of A and $A^* \subseteq A_i(d^*)$. Then $A^* \subseteq N$.

Fix $\gamma \in A^*(\kappa_i)$. In N, for each μ, define

$$\bar{D}_\mu = \{(g, r, h^1) \geq_{\mathbb{R}_i \times Col(s_i(\gamma)^{+3}, < \kappa_i)} (f \oplus \mu, q' \setminus (i+1), H^0(\gamma)) \mid \text{there are } t \geq (q' \restriction i)_\mu, h^0 \geq (H^0)(\mu \restriction d) \text{ with } t^\frown \langle g, h^0, h^1 \rangle^\frown r \in D\}.$$

Clearly $\bar{D}_\mu \in N$ is open dense in $\mathbb{R}_i \times Col(s_i(\gamma)^{+3}, < \kappa_i)$ above $(f \oplus \mu, q' \setminus (i+1), H^0(\gamma))$. We now define D_γ as the collection of $(g, r) \in \mathbb{R}^*_i$ such that there is $(h^1)^* \geq (H^1_i)(\gamma)$ satisfying the following requirement: for each $\mu \in A^*$ with $\mu(\kappa_i) = \gamma$,

- $\mathrm{dom}(\mu) \subseteq \mathrm{dom}(g)$.
- for all $t \geq^* (q' \restriction i)_\mu$, $h^0 \geq H^0(\gamma)$, if there are $g' \geq g \oplus \mu$, $h^1 \geq (h^1)^*$ $a \in [\{\xi \mid i < \xi < \kappa\}]^{n-1}$, $\vec{\tau} \in \prod_{\beta \in a} A^r_\beta$, and $r' \geq^* r + \vec{\tau}$ such that
$$t^\frown \langle g', h^0, h^1 \rangle^\frown r' \in \bar{D}_\mu,$$
then there is a $(r, n-1)$-fat tree T such that for every maximal $\vec{\tau} \in T$,
$$t^\frown \langle g \oplus \mu, h, (h^1)^* \rangle^\frown (r + \vec{\tau}) \in \bar{D}_\mu.$$

By our induction hypothesis as in Remark 3.19 (we apply the remark with $\mathbb{P}_{\vec{E} \setminus (i+1)}$), the property of p^{n-1}, and the fact that the number of such t and h^0 is at most $s_i(\gamma)^{+2}$, which is below the closure of $\mathbb{R}^*_i$ and $Col(s_i(\gamma)^{+3}, < \kappa_i)$, we have that D_γ is open dense in N. Hence, $(f^*, r^*) \in D_\gamma$, we obtain a witness $(h^1)^* =: (h^1)^\gamma$. Let $(H^1)^*(\gamma) = (h^1)^\gamma$. Let q^i be such that $q^i \restriction i = q' \restriction i$, $q^i_i = \langle f^*, A^*, (H^0)^*, (H^1)^* \rangle^\frown r^*$, where, $(H^0)^*(\tau) = (H^0)(\tau \restriction d)$. Recall that we take q^* as the least $\leq^*$-upper bound of $\langle q^i \mid i < \kappa \rangle$. We have that q^* has the follow property: Fix $i < \kappa$ and $\mu \in A^{q^*}_i$. Then,

- either for every $t \geq (q^* \restriction i)_\mu$, $h^0 \geq (H^0_i)^{q^*}$, $a \in [\{\xi \mid i < \xi < \kappa\}]^{n-1}$, and $\vec{\tau} \in \prod_{\beta \in a} A^{q^*}_\beta$, we have that $t^\frown \langle f^{q^*}_i \oplus \mu, h^0, (H^1_i)^{q^*}(\mu(\kappa_i)) \rangle^\frown (q^* \setminus (i+1)) \notin D$,

- or there are $t \geq (q^* \restriction i)_\mu$, $h^0 \geq (H_i^0)^{q^*}$, and a fat tree T of height $n-1$ (not necessarily fully compatible with q^*) such that for every $\vec{\tau} \in T$ maximal, $t^\frown \langle f_i^{q^*} \oplus \mu, h^0, (H_i^1)^{q^*}(\mu(\kappa_i))^\frown \vec{\tau} \in D$.

The reason that in the latter case, T might not be fully compatible with q^* is that by the property of $D_{\mu \restriction d_i^{q^i}}$, there is an $(n-1, q^* \setminus (i+1))$-fat tree fully compatible with q^* such that for each $\vec{\tau}$ maximal in the tree, there are witnesses $t =: t_{\vec{\tau}}$ and $h^0 =: h^0_{\vec{\tau}}$. We then use Lemma 3.14 to shrink the fat tree to get the fixed t and h^0.

Step B By the Prikry property (see Remark 3.17) we have a possibility to choose $s \geq^* (q^* \restriction i)_\mu$ and $h^0 \geq (H_i^0)^{q^*}$ so that we have the following two cases.

Case 1: For all $i < \kappa$, the collection B_i of $\mu \in OB_i(d_i^{q^*})$ such that "for every $t \geq^* (q^* \restriction i)_\mu$, $h^0 \geq (H_0^i)^{q^*}(\mu)$, $a \in [\{\xi \mid i < \xi < \kappa\}]^{n-1}$ and $\vec{\tau} \in \prod_{\beta \in a} A_\beta^{q^*}$ with $t^\frown \langle f_i^{q^*} \oplus \mu, h, (H_i^i)^{q^*}(\mu(\kappa_i))\rangle^\frown (q^* \setminus (i+1)) + \vec{\tau} \notin D$" is of measure-one. For this case, let p^n be obtained from q^* by shrink $A_i^{q^*}$ to B_i.

Case 2: There is $i < \kappa$ such that the collection of B_i of μ such that "there are $t \geq^* (q^* \restriction i)_\mu$, $h^0 \geq (H_0^i)^{q^*}(\mu)$, and a $(n-1, q^* \setminus (i+1))$-fat tree T such that for each $\vec{\tau} \in T$ maximal, $t(\mu)^\frown f_i^{q^*} \oplus \mu, h^0, (H_i^1)^{q^*}(\mu(\kappa_i))\rangle^\frown (q^* \setminus (i+1)) + \vec{\tau} \in D$" is of measure-one. Assume i is the least such. For each μ, let $t = t(\mu)$ and $h^0 = h^0(\mu)$ and $T = T(\mu)$ be the witnesses for the property. Use Lemma 3.15 to obtain $p^n \geq^* q$ so that for every $\tau \in A_i^{p^n}$, $(p^n \restriction i)_\tau = t(\tau \restriction d_i^{q^*})$, and there is a $(n-1, p^n)$-fat tree $T^*(\tau)$ which projects down to a subtree of $T(\tau \restriction d_i^{q^*})$. Let T be such that $Level_0(T) \in E_i(d_i^{p^n})$, $Level_0(T)$ projects down to a subset of B_i, and for $\tau \in Level_0(T)$, $T_\tau = T^*(\tau)$. Shrink all relevant measure-one sets in p^n so that all relevant objects appear in T, and finally, shrink T to be fully compatible with p^n.

We conclude the following property of p^n: if there is a direct extension of an n-step extension of p^n entering D, then there is a (p^n, n)-fat tree T which is compatible with p^n such that for every maximal node $\vec{\tau} \in T$, $p^n + \vec{\tau} \in D$.

Now, let p^* be a $\geq^*$-upper bound of $\langle p^n \mid n < \omega\rangle$. If $q \geq p^*$ and $q \in D$, then $q \geq^* p^* + \vec{\tau}$ for some ρ. Say $|\vec{\tau}| = n$. Assume that $n > 1$ (the case $n = 1$ is slightly simpler). This implies that $q \geq^* p^* + \vec{\tau} \geq p^n + \vec{\tau}'$, where $\vec{\tau}'$ is obtained by restricting functions in $\vec{\tau}$ properly. This implies that there is an (p^n, n)-tree T such that for every $\vec{\mu} \in T$ maximal, $p^n + \vec{\mu} \in D$. Let T^* be a pullback of T so that T^* is fully compatible with p^*. Then for every $\vec{\mu} \in T^*$ maximal, $p^* + \vec{\mu} \in D$.

$\square$

Corollary 3.20. $(\mathbb{P}_{\vec{E}}, \leq, \leq^*)$ *has the Prikry property. Namely for each forcing statement φ and a condition p, there is $p^* \geq^* p$ that either $p^* \Vdash \varphi$ or $p^* \Vdash \neg\varphi$.*

Proof. Let $D = \{q \mid q \Vdash \varphi \text{ or } q \Vdash \neg\varphi\}$. Let $p^* \geq^* p$ and a fat tree T fully compatible with p^* witnessing the strong Prikry property for D. By shrinking measure-one sets and the fat tree as in Lemma 3.14, we may assume that either for all maximal $\vec{\mu} \in T$, $p + \vec{\mu} \Vdash \varphi$, or for all maximal $\vec{\mu} \in T$, $p + \vec{\mu} \Vdash \varphi$. Let $q \geq p^*$ be such that q decides φ. Without loss of generality, assume $q \Vdash \varphi$. Extend if necessary, assume $q \geq p^* + \vec{\mu}$ for some maximal $\vec{\mu} \in T$. This means that for every maximal $\vec{\tau} \in T$, $p^* + \vec{\tau} \Vdash \varphi$. Since $\{p^* + \vec{\tau} \mid \vec{\tau} \in T \text{ is maximal}\}$ is predense above p^*, we have that $p^* \Vdash \varphi$. $\square$

Following the standard arguments of factorization, Prikry property, and the strong Prikry property, we have the following cardinal arithmetic

Theorem 3.21. *After forcing with $\mathbb{P}_{\langle E_i \mid i<\kappa\rangle}$, we have that*

1. *κ is the first inaccessible cardinal.*
2. *GCH holds for every regular cardinal below κ. Each singular cardinal below κ is a strong limit, and SCH fails for every singular cardinal below κ.*
3. *for every singular cardinal $\lambda < \kappa$, $\mathfrak{u}_\lambda = \lambda^+ < \lambda^{++} = 2^\lambda$.*

Proof. (1) Note that κ is strong limit. If κ is singular, then let $p \in \mathbb{P}_{\vec{E}}$, $\alpha < \kappa$, and $\dot{f}$ be a $\mathbb{P}$-name such that $p \Vdash \dot{f} : \alpha \to \kappa$ is cofinal. Extend p if necessary, assume $\alpha + 1 \in \mathrm{Supp}(p)$. Forcing above p factors to $\mathbb{P}_0 \times \mathbb{P}_1$ where $|\mathbb{P}_0| = s_{\alpha+1}(f^p_{\alpha+1}(\kappa_{\alpha+1}))^{+2}$ and $(\mathbb{P}_1, \leq^*)$ is $s_{\alpha+1}(f^p_{\alpha+1}(\kappa_{\alpha+1}))^{+3}$-closed. By the Prikry property and by theo closure, we can find $q \geq^* p \restriction \mathbb{P}_1$ such that for each $\gamma < \alpha$, there is a maximal antichain $A_\gamma \subseteq \mathbb{P}_0$ above $p \restriction \mathbb{P}_0$ such that for every $r \in A_\gamma$, $r^\frown q$ decides $\dot{f}(\gamma)$. In V, let $X = \{\xi \mid \exists\gamma\exists r \in A_\gamma(r^\frown q \Vdash \dot{f}(\gamma) = \xi)\}$. Then $|X| < \kappa$ and $(p \restriction \mathbb{P}_0)^\frown q \Vdash \mathrm{rng}(\dot{f}) \subseteq X$, which is a contrau

(2) Follow the same analysis as in [14].

(3) The argument is similar to Theorem 2.17 except that we apply the version of the strong Prikry property in this section.

$\square$

Remark 3.22. Since the forcing is κ^{++}-c.c., all cardinals above and including κ^{++} are preserved. One can follow the argument in [1] to show that κ^+ is also preserved.

Corollary 3.23. *It is consistent that GCH holds at every regular and SCH fails at every singular λ while $\mathfrak{u}_\lambda = \lambda^+$.*

Proof. From the previous model $V^{\mathbb{P}_{\langle E_i \mid i<\kappa\rangle}}$, just take $M = (V^{\mathbb{P}_{\langle E_i \mid i<\kappa\rangle}})_\kappa$ which is a *ZFC* model (by inaccessibility of κ) which exhibit the corollary. $\square$

Corollary 3.24. *It is consistent that an inaccessible κ satisfy $\mathfrak{u}_\kappa > \kappa^+$ while there is club $C \subseteq \kappa$ such that for every $\lambda \in C$, $\mathfrak{u}_\lambda = \lambda^+ < 2^\lambda$.*

Proof. From the model $V^{\mathbb{P}_{\langle E_i | i<\kappa\rangle}}$ force with $Add(\kappa, \kappa^{++})$, then by the κ-closure of the forcing we did not change $\mathfrak{u}_\lambda$ for $\lambda < \kappa$ and $\mathfrak{u}_\kappa = \kappa^{++}$ in the extension. $\square$

4. Open problems

Question 4.1. What is $\mathfrak{u}_\kappa$ in the model of theorem 3.21?

References

1. Omer Ben-Neria and Jing Zhang, *Approximating diamond principles on products at an inaccessible cardinal*, Submiited (2022).
2. A.D. Brooke-Taylor, V. Fischer, S.D. Friedman, and D.C. Montoya, *Cardinal characteristics at κ in a small $\mathfrak{u}(\kappa)$ model*, Annals of Pure and Applied Logic **168** (2017), no. 1, 37–49.
3. Eduard Cech, *On bicompact spaces*, Annals of Mathematics **38** (1937), no. 4, 823–844.
4. James Cummings, *Iterated Forcing and Elementary Embeddings*, pp. 775–883, Springer Netherlands, Dordrecht, 2010.
5. James Cummings and Charles G. Morgan, *Ultrafilters on singular cardinals of uncountable cofinality*, arXiv: Logic (2019), preprint.
6. Shimon Garti, Moti Gitik, and Saharon Shelah, *Cardinal characteristics at aleph omega*, Acta. Mathematica Hungarica **160** (2020), 320–336.
7. Shimon Garti, Menachem Magidor, and Saharon Shelah, *On the spectrum of characters of ultrafilters*, Notre Dame J. Formal Log. **59** (2016), 371–379.
8. Shimon Garti and Saharon Shelah, *A strong polarized relation*, The Journal of Symbolic Logic **77** (2012), no. 3, 766–776.
9. Shimon Garti and Saharon Shelah, *The ultrafilter number for singular cardinals*, Acta. Mathematica Hungarica **137** (2012), 296–301.
10. Moti Gitik, *Prikry Type Forcings*, pp. 1351–1447, Springer Netherlands, Dordrecht, (2010).
11. ______, *A uniform ultrafilter over a singular cardinal with a singular character*, Acta Math. Hungar **162** (2020), 325–332.
12. Moti Gitik and Saharon Shelah, *On densities of box products*, Topology and its Applications **88** (1996), 219–237.
13. C. Ward Henson, *Foundations of nonstandard analysis*, pp. 1–49, Springer Netherlands, Dordrecht, 1997.
14. Sittinon Jirattikansakul, *Blowing up the power of singular cardinal of uncountable cofinality with collapses*, preprint (2020), arXiv:2011.00409.
15. Péter Komjáth and Vilmos Totik, *Ultrafilters*, The American Mathematical Monthly **115** (2008), no. 1, 33–44.
16. Kenneth Kunen, *Introduction to independence proofs*, North-Holand, 1980.
17. Menachem Magidor, *On the singular cardinals problem ii*, Annals of Mathematics **106** (1977), no. 3, 517–547.
18. Carmi Merimovich, *Extender-based radin forcing*, Transactions of the American Mathematical Society **355** (2003), no. 5, 1729–1772.
19. Lon Berk Radin, *Adding Closed Cofinal Sequences to Large Cardinals*, Annals of Mathematical Logic **22** (1982), no. 3, 243—-261.
20. Dilip Raghavan and Saharon Shelah, *A Small Ultrafilter Number at Smaller Cardinals*, Archive for Mathematical Logic **59** (2020), 325–334.
21. M. H. Stone, *Applications of the theory of boolean rings to general topology*, Transactions of the American Mathematical Society **41** (1937), no. 3, 375–481.